

# Heat trace asymptotics with transmittal boundary conditions and quantum brane-world scenario

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## Abstract

We study the spectral geometry of an operator of Laplace type on a manifold with a singular surface. We calculate several first coefficients of the heat kernel expansion. These coefficients are responsible for divergences and conformal anomaly in quantum brane-world scenario.

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## 1 Motivations

It is well known that the regularized one-loop effective action in Euclidean quantum field theory is given by the following formal expression

$$W^{\text{reg}} = \frac{1}{2} \log \det(D)_{\text{reg}} = -\frac{\mu^{2s}}{2} \int_0^\infty dt \, t^{s-1} \text{Tr}(\exp(-tD)), \quad (1)$$

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where we have introduced the (zeta-function) regularization parameter  $s$ , which should be set to zero after calculations. The parameter  $\mu$  of the dimension of mass makes the effective action (1) dimensionless for any  $s$ . The value of  $\mu$  is to be fixed by a normalization condition. The operator  $D$  is a partial differential operator which appears in the quadratic part of the classical action. We assume that  $D$  is a second order operator of Laplace type and that there is an asymptotic series

$$\mathrm{Tr}(f \exp(-tD)) \cong \sum_{n=0}^{\infty} t^{\frac{n-m}{2}} a_n(f, D) \quad (2)$$

as  $t \downarrow 0$ . Here  $m$  is the dimension of the underlying manifold  $M$  and  $f$  is a smearing (or localizing) function. Near  $s = 0$  the regularized effective action behaves as

$$W^{\mathrm{reg}} \cong -\frac{1}{2s} a_m(1, D) + O(s^0). \quad (3)$$

Therefore, the heat kernel coefficient  $a_m$  provides complete information on the one-loop divergences. In most of the cases that one considers, the coefficients  $a_n$  are locally computable; equivalently, this means that the counter-terms are local. If the operator  $D$  is conformally covariant, then  $a_m$  also defines the trace anomaly in the stress-energy tensor.

The heat kernel asymptotics on (smooth) manifolds with or without a boundary have been studied in some detail. Relatively less is known about the case when there are some kinds of “non-smoothness” inside the manifold. Only the cases of point-like singularities, either conical [15, 14, 16, 19, 17] or delta-function ones [1], have attracted considerable attention. We also mention a related work [34].

In the present paper we deal with the heat kernel asymptotics for the case when the operator  $D$  has a “non-smoothness” on a surface  $\Sigma$  of co-dimension one. Such kind of singularities appear in many problems of quantum field theory as, e.g. the Casimir energy calculations. The case when the metric is smooth across  $\Sigma$  has been studied recently by Bordag and Vassilevich [9] and by Moss [29]. In the present paper we allow normal derivatives of the metric to jump on  $\Sigma$ . This study is motivated by (and has applications in) the brane-world scenario [32, 33] which operates with the metric of the type

$$(ds)^2 = (dx^5)^2 + e^{-\alpha|x^5|} (ds_4)^2, \quad (4)$$

where  $\alpha$  is a constant and where  $(ds_4)^2$  is a line element on four-dimensional hypersurface. Due to the presence of the absolute value of the 5th coordinate in (4), the normal derivative of the metric jumps on the surface  $\Sigma$  defined by the vanishing of the coordinate  $x^5$ . It is also assumed that the bulk action is supplemented by a surface term concentrated on  $\Sigma$ . This model can be further generalized to allow for a more general line element and a more general singular surface  $\Sigma$ . One can also imagine a similar construction in dimension  $m$  other

than 5, though the codimension of  $\Sigma$  will be always supposed to be 1. It is clear that the quadratic part of the classical matter action for a quite general class of the brane-world models should be of the form

$$S_2 = \int_M d^5x \sqrt{g} \phi D \phi, \quad (5)$$

where  $\phi$  describes the bulk field fluctuations, and the operator  $D$  is<sup>1</sup>

$$D = -(\nabla^2 + E(x)) + U \delta_\Sigma. \quad (6)$$

Here  $\nabla$  is a suitable covariant derivative, and  $E(x)$  and  $U(x)$  are endomorphisms (matrix valued fields). Let  $h$  be the determinant of the induced metric on  $\Sigma$ . Then  $\delta_\Sigma$  is a delta function defined such that

$$\int_M dx \sqrt{g} \delta_\Sigma f(x) = \int_\Sigma dx \sqrt{h} f(x). \quad (7)$$

We shall assume that  $D$  is smooth on  $M - \Sigma$ . On the hypersurface  $\Sigma$ , we shall only assume that the leading symbol (metric) of  $D$  is continuous; the normal derivatives of the metric are not assumed to be continuous on  $\Sigma$ . Furthermore, we shall impose no assumption of continuity on the remaining tensors ( $E$ , curvature, etc.) on  $\Sigma$ .

Let  $x^m$  be a smooth function so the equation  $x^m = 0$  defines the hypersurface  $\Sigma$  and so  $dx^m \neq 0$  on  $\Sigma$ . It is convenient to introduce a coordinate system on  $M$  such that in a neighbourhood of  $\Sigma$

$$(ds)^2 = (dx^m)^2 + g_{ab} dx^a dx^b. \quad (8)$$

The spectral problem for  $D$  on  $M$  as it stands is ill-defined owing to the discontinuities (or singularities) on  $\Sigma$ . It should be replaced by a pair of spectral problems on the two sides  $M^\pm$  of  $\Sigma$  together with suitable matching conditions on  $\Sigma$ . In order to find such matching conditions, we consider an eigenfunction  $\phi_\lambda$  of the operator (6):

$$D \phi_\lambda = \lambda \phi_\lambda. \quad (9)$$

It is clear that  $\phi_\lambda$  must be continuous on  $\Sigma$ :

$$\phi|_{x^m=+0} = \phi|_{x^m=-0}. \quad (10)$$

Otherwise, the second normal derivative of  $\phi_\lambda$  would create a  $\delta'$  singularity on  $\Sigma$  which is absent on the right hand side of (9). Let us integrate (9) over a small cylinder  $\mathcal{C} = C^{m-1} \times [-\epsilon, +\epsilon]$

$$\int_{\mathcal{C}} d^m x \sqrt{g} \left( -\nabla_m^2 \phi_\lambda - [\nabla_a^2 \phi_\lambda + (E + \lambda) \phi_\lambda] \right) + \int_C d^{m-1} x \sqrt{h} U \phi_\lambda = 0. \quad (11)$$

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<sup>1</sup>Note that in the present paper we neglect possible derivative terms in the surface action for simplicity

We now take the limit as  $\epsilon \rightarrow 0$ . Since the expression in the square brackets in (11) is bounded, the contribution that this term makes vanishes in the limit. We obtain

$$0 = \int_C d^{m-1}x \sqrt{h} (-\nabla_m \phi_\lambda|_{x^m=+0} + \nabla_m \phi_\lambda|_{x^m=-0} + U\phi_\lambda) . \quad (12)$$

Since  $C$  and  $\lambda$  are arbitrary, we conclude that a proper matching condition for the normal derivatives is

$$-\nabla_m \phi|_{x^m=+0} + \nabla_m \phi|_{x^m=-0} + U\phi = 0 . \quad (13)$$

A more mathematically careful construction of these transmittal boundary conditions will be given in subsequent sections.

There have been already many works devoted to the quantization of bulk fields<sup>2</sup> in the brane-world scenario (see e.g. [20, 35, 18, 23, 31, 2, 13, 24]). However, the heat kernel expansion, divergences and renormalization have not been discussed to a considerable order of generality.

Here is a brief guide to this paper; a more expanded discussion is given in Section 2 after the necessary notation has been introduced. In Section 2, we give a more precise statement of transmittal boundary conditions and discuss the geometry of operators of Laplace type. In section 3 we consider a smooth structure and the gluing construction. The invariance theory is developed in section 4. Section 5 deals with reduction of the transmittal problem to Dirichlet and Neumann boundary value problems. In section 6, we construct a transmittal problem for the de Rham complex. We use this problem to complete the calculation of several first heat kernel coefficients. The coefficient  $a_4$  is calculated in section 7. In section 8 we calculate  $a_5$  for a restricted class of transmittal problems and discuss applications to the brane-world scenario. Appendix contains some technical details.

## 2 Introduction

Let  $\Sigma$  be a codimension 1 hypersurface of a compact smooth manifold which divides  $M$  into two manifolds  $M^\pm$ . This means that

$$M := M^+ \cup_\Sigma M^-$$

is the union of two compact manifolds  $M^\pm$  along their common boundary  $\Sigma$ . We assume given a Riemannian metric which is continuous on  $M$  and smooth when restricted to  $M^\pm$ . Let  $V$  be a smooth vector bundle over  $M$  and let  $D^\pm$  be operators of Laplace type on  $V^\pm := V|_{M^\pm}$ ; no further conditions are placed on  $D^\pm$  apart from the assumption that the leading symbols agree on  $\Sigma$ . The operators

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<sup>2</sup>Not to be mixed with quantum effects of the so-called “brane matter” which is confined on the singular surface

$D^\pm$  determine canonical connections  ${}^\pm\nabla$  on  $V^\pm$ , see equation (16) below. Let  $U$  be an auxiliary endomorphism of  $V_\Sigma := V|_\Sigma$ . Let  $\nu$  be the inward unit normal of  $\Sigma \subset M^+$  and let  $\phi := (\phi^+, \phi^-)$  be a pair of smooth sections to  $V^\pm$ . We define the *transmittal operator*

$$\mathcal{B}_U\phi = \{\phi^+|_\Sigma - \phi^-|_\Sigma\} \oplus \{(\nabla_\nu^+\phi^+)|_\Sigma - (\nabla_\nu^-\phi^-)|_\Sigma - U\phi^+|_\Sigma\}. \quad (14)$$

An elliptic boundary condition for a  $q^{th}$  order operator on a vector bundle of dimension  $r$  must involve  $\frac{1}{2}qr$  conditions. We set  $q = 2$  as we are considering operators of Laplace type. Neumann and Dirichlet boundary conditions involve  $\frac{1}{2}2r = r$  conditions. Transmittal boundary conditions fulfil this counting condition; since we have two vector bundles  $V^\pm$ , we must specify  $\frac{1}{2}2(2r) = 2r$  conditions which is what the vanishing of the operator in equation (14) imposes:

$$\phi^+|_\Sigma = \phi^-|_\Sigma \text{ and } \nabla_\nu^+\phi^+|_\Sigma = \{\nabla_\nu^-\phi^-|_\Sigma\} + U\{\phi^+|_\Sigma\}.$$

Let  $D := (D^+, D^-)$  act on  $\phi := (\phi^+, \phi^-)$  in the natural fashion. We restrict the domain of  $D$  to pairs  $\phi$  so that  $\mathcal{B}_U\phi = 0$ . Let  $D_{\mathcal{B}_U}$  be the realization of  $D$  on this domain and let  $e^{-tD_{\mathcal{B}_U}}$  be the associated fundamental solution of the heat equation. Let  $f = (f^+, f^-)$  where the  $f^\pm$  are smooth on  $M^\pm$  and where  $f^+|_\Sigma = f^-|_\Sigma$ ; no matching is assumed for the normal derivatives of  $f$ . Let

$$a(f, D, U)(t) := \text{Tr}_{L^2}\{f e^{-tD_{\mathcal{B}_U}}\}$$

be the heat trace. If the  $D^\pm$  are formally self-adjoint, and if  $U$  is self-adjoint, then  $D_{\mathcal{B}_U}$  self-adjoint. Thus we can find a discrete spectral resolution  $\{\lambda_i, \phi_i\}$  where the  $\{\phi_i\}$  form a complete orthonormal basis for  $L^2(V)$ , where  $D^\pm\phi_i^\pm = \lambda_i\phi_i^\pm$ , and where  $\mathcal{B}_U\phi = 0$ . We then have:

$$a(f, D, U)(t) = \sum_i e^{-t\lambda_i} \int_M f(\phi_i, \phi_i). \quad (15)$$

**Assumption 2.1** *There exists a full asymptotic series as  $t \downarrow 0$ :*

$$a(f, D, U)(t) \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(f, D, U)$$

where the heat trace coefficients  $a_n(f, D, U)$  are locally computable, i.e. there are local invariants  $a_n(x^\pm, D^\pm)$  defined on  $M^\pm$  and local invariants  $a_n^\Sigma(y, f, D, U)$  defined on  $\Sigma$  so that:

$$\begin{aligned} a_n(f, D, U) &= a_n^+(f, D) + a_n^-(f, D) + a_n^\Sigma(f, D, U) \text{ where} \\ a_n^\pm(f, D) &= \int_{M^\pm} f(x^\pm) a_n(x^\pm, D^\pm) \text{ and} \\ a_n^\Sigma(f, D, U) &= \int_\Sigma a_n^\Sigma(y, f, D, U). \end{aligned}$$

We remark that Assumption 2.1 has been established by [9, 29] if the leading symbol (i.e. the metric) is smooth.

Before discussing the interior invariants  $a_n^\pm$ , we must describe the geometry of operators of Laplace type. The operators  $D^\pm$  determine natural connections  $\nabla^\pm$  and natural  $0^{th}$  order operators  $E^\pm$  so that

$$D^\pm = -\{\text{Tr}(\nabla^\pm \nabla^\pm) + E^\pm\}.$$

If we choose a system of local coordinates and a local frame, we can express:

$$D^\pm = -(g^{\pm, \mu\nu} \partial_\mu \partial_\nu + A^{\pm, \mu} \partial_\mu + B^\pm)$$

where we adopt the Einstein convention and sum over repeated indices. Let  $\Gamma^\pm$  be the Christoffel symbols of the metrics  $g^\pm$ . The connection 1 forms  $\omega^\pm$  of  $\nabla^\pm$  and the endomorphisms  $E^\pm$  are then given by

$$\begin{aligned} \omega_\delta^\pm &= \frac{1}{2} g_{\nu\delta}^\pm (A^{\pm, \nu} + g^{\pm, \mu\sigma} \Gamma_{\mu\sigma}^\pm \nu I) \text{ and} \\ E^\pm &= B^\pm - g^{\pm, \nu\mu} (\partial_\nu \omega_\mu^\pm + \omega_\nu^\pm \omega_\mu^\pm - \omega_\sigma^\pm \Gamma_{\nu\mu}^\pm \sigma); \end{aligned} \quad (16)$$

see [22] for further details. Let indices  $i, j, k$ , and  $l$  range from 1 to  $m$  and index a local orthonormal frame for the tangent bundle of the manifold. Let  $R_{ijkl}^\pm$  be the components of the curvature tensor of the Levi-Civita connection; with our sign convention the Ricci tensors  $\rho^\pm$  and the scalar curvatures  $\tau^\pm$  are given by:

$$\rho_{ij}^\pm := R_{ikkk}^\pm \text{ and } \tau^\pm := \rho_{ii} = R_{ijji}.$$

Let  $\Omega_{ij}^\pm$  be the components of the curvature tensors of the connection  $\nabla^\pm$ . The interior invariants have been computed previously in the smooth context. They vanish if  $n$  is odd and have been determined explicitly for  $n = 0, 2, 4, 6, 8, 10$  - see for example [3, 4, 21, 36]. The presence of the junction discontinuity along  $\Sigma$  does not affect the interior invariants  $a_n^\pm$  and consequently we may apply these results to see that:

**Theorem 2.2** *The invariants  $a_n^\pm$  vanish if  $n$  is odd. We have:*

1.  $a_0^\pm(f, D) = (4\pi)^{-m/2} \int_{M^\pm} f \text{Tr}(I).$
2.  $a_2^\pm(f, D) = (4\pi)^{-m/2} \frac{1}{6} \int_{M^\pm} f \text{Tr}(\tau^\pm I + 6E^\pm).$
3.  $a_4^\pm(f, D) = (4\pi)^{-m/2} \frac{1}{360} \int_{M^\pm} f \text{Tr} \{60E_{;kk}^\pm + 60R_{ijji}^\pm E^\pm + 180E^\pm E^\pm + 30\Omega_{ij}^\pm \Omega_{ij}^\pm + (12\tau_{;kk}^\pm + 5(\tau^\pm)^2 - 2|(\rho^\pm)^2| + 2|(R^\pm)^2|)I\}.$

We now introduce some additional notation to describe the invariants  $a_n^\Sigma$ . Let indices  $a, b, c$ , and  $d$  index a local orthonormal frame  $\{e_a\}$  for the tangent bundle of  $\Sigma$ ; we complete this frame to a frame for the tangent bundle of  $M$  by letting

$e_m := \nu$  be the inward unit normal of  $\Sigma \subset M^+$ . Let  $\nu^\pm := \pm \nu$  be the inward unit normals of  $\Sigma \subset M^\pm$  and let

$$L_{ab}^\pm := (\nabla_{e_a}^\pm e_b, \nu^\pm)|_\Sigma$$

be the associated second fundamental forms. Let

$$\omega_a := \nabla_a^+ - \nabla_a^-.$$

Since the difference of two connections is tensorial,  $\omega_a$  is a well defined endomorphism of  $V_\Sigma$ . The tensor  $\omega_a$  is *chiral*; it changes sign if the roles of  $+$  and  $-$  are reversed. Since we can describe the matching condition on the normal derivatives in the form:

$$(\nabla_{\nu^+}^+ \phi^+)|_\Sigma + (\nabla_{\nu^-}^- \phi^-)|_\Sigma = U \phi|_\Sigma,$$

the tensor field  $U$  is *non-chiral* as it is not sensitive to the roles of  $+$  and  $-$ . The main result of this paper is the following Theorem which determines the invariants  $a_n^\Sigma$  for  $n = 0, 1, 2, 3$ ; the invariant  $a_4^\Sigma$  is a bit more combinatorially complex and the formula for this invariant is discussed in Section 7.

### Theorem 2.3

1.  $a_0^\Sigma(f, D, U) = 0$ .
2.  $a_1^\Sigma(f, D, U) = 0$ .
3.  $a_2^\Sigma(f, D, U) = (4\pi)^{-m/2} \frac{1}{6} \int_\Sigma \text{Tr} \{2f(L_{aa}^+ + L_{aa}^-)I - 6fU\}$ .
4.  $a_3^\Sigma(f, D, U) = (4\pi)^{(1-m)/2} \frac{1}{384} \int_\Sigma \text{Tr} \{ \frac{3}{2}f(L_{aa}^+ L_{bb}^+ + L_{aa}^- L_{bb}^- + 2L_{aa}^+ L_{bb}^-)I$   
 $+ 3f(L_{ab}^+ L_{ab}^+ + L_{ab}^- L_{ab}^- + 2L_{ab}^+ L_{ab}^-)I + 9(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+}^+ + f_{;\nu^-}^-)I$   
 $+ 48fU^2 + 24f\omega_a \omega_a - 24f(L_{aa}^+ + L_{aa}^-)U - 24(f_{;\nu^+}^+ + f_{;\nu^-}^-)U \}.$

We can now give a more complete outline to the paper than was given in the introduction. In Section 3, we give an alternate formulation of transmittal boundary conditions in terms of  $C^1$  structures that will be convenient when considering conformal variations. In Section 4, we use invariance theory and dimensional analysis to prove the following result which gives the general form that the invariants  $a_n^\Sigma$  have:

**Lemma 2.4** *There exist universal constants so that:*

1.  $a_0^\Sigma(f, D, U) = 0$ .
2.  $a_1^\Sigma(f, D, U) = \int_\Sigma c_1 f \text{Tr}(I)$
3.  $a_2^\Sigma(f, D, U) = (4\pi)^{-m/2} \frac{1}{6} \int_\Sigma \text{Tr} \{d_1 f(L_{aa}^+ + L_{aa}^-)I + d_2(f_{;\nu^+}^+ + f_{;\nu^-}^-)I + d_3 fU\}$ .

$$\begin{aligned}
4. \quad a_3^\Sigma(f, D, U) = & (4\pi)^{(1-m)/2} \frac{1}{384} \int_\Sigma \text{Tr} \{ c_2(L_{aa}^+ L_{bb}^-) I + c_3(L_{ab}^+ L_{ab}^-) I \\
& + c_4(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+}^+ - f_{;\nu^-}^-) I + c_5(f_{;\nu^+ \nu^+}^+ + f_{;\nu^- \nu^-}^-) I \\
& + c_6(E^+ + E^-) + c_7(R_{ijji}^+ + R_{ijji}^-) I + c_8(\rho_{mm}^+ + \rho_{mm}^-) I \\
& + d_4 f(L_{aa}^+ L_{bb}^+ + L_{aa}^- L_{bb}^- + 2L_{aa}^+ L_{bb}^-) I + d_5 f(L_{ab}^+ L_{ab}^+ + L_{ab}^- L_{ab}^- + 2L_{ab}^+ L_{ab}^-) I \\
& + d_6(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+}^+ + f_{;\nu^-}^-) I + d_7 f U^2 + d_8 f(L_{aa}^+ + L_{aa}^-) U \\
& + d_9(f_{;\nu^+}^+ + f_{;\nu^-}^-) U + e_1 f \omega_a \omega_a \}.
\end{aligned}$$

If we suppose that the operator  $D$  is smooth and that the localizing function  $f$  is smooth on all of  $M$ , then  $\Sigma$  plays no role and thus the invariants  $a_n^\Sigma$  vanish. We use this observation to show in Lemma 4.1 that the coefficients  $c_i$  must vanish. In Section 5 we recall formulas for the heat trace invariants on manifolds with boundary; see Lemma 5.1. We use these formulas to determine the coefficients  $d_j$ , see Lemma 5.3 for details. In Section 6, we construct a transmittal problem for the de Rham complex and use the resulting local index theorem to show that the one remaining unknown coefficient has the value  $e_1 = 24$ ; this completes the proof of Theorem 2.3. We remark that Moss [29] used different methods to show that  $e_1 = 24$ . In Section 7, we perform a similar analysis to determine the invariant  $a_4^\Sigma$ . The value of the coefficients  $c_1, d_1, d_2, d_3, c_5, c_6, c_7$ , and  $c_8$  agrees with the values calculated previously in [9] using other methods.

### 3 Glueing constructions

We use the geodesic flow to identify a neighborhood of  $\Sigma$  in  $M^+$  with  $\Sigma \times [0, \varepsilon)$  and a neighborhood of  $\Sigma$  in  $M^-$  with  $\Sigma \times (-\varepsilon, 0]$  for some  $\varepsilon > 0$  so that the curves  $t \rightarrow (y, t)$  are unit speed geodesics normal to the boundary  $\Sigma := \Sigma \times \{0\}$ . We define a canonical smooth structure on  $M = M^+ \cup M^-$  by glueing along  $\Sigma \times \{0\}$ . Note that the metric then takes the form:

$$^\pm ds^2 = g_{ab}^\pm(y, t) dy^a \circ dy^b + dt \circ dt.$$

We can use  $U$  to define a canonical  $C^1$  structure on  $V$ . Let  $s_\Sigma$  be a local frame for  $V|_\Sigma$ . We use parallel transport along the geodesic normals to define a local frame  $s^-$  for  $V^-$  near  $\Sigma$  so  $\nabla_\nu s^- = 0$ . We twist a corresponding parallel frame over  $V^+$  to define a local frame  $s^+$  for  $V^+$  near  $\Sigma$  so  $\nabla_\nu s^+ = U s^+$ . We glue  $s^+$  to  $s^-$  over  $\Sigma$  to define a  $C^1$  structure for  $V$  over  $M$  which is characterized by the property that  $\nabla_\nu^+ - \nabla_\nu^- = U$ . We then have that  $\mathcal{B}_U \phi = 0$  if and only if  $\phi \in C^1(V)$ . When studying variations of the form  $D(\varepsilon) := e^{\varepsilon f} D$  we will fix the  $C^1$  structure or equivalently choose  $U(\varepsilon)$  so the transmittal boundary condition  $\mathcal{B}_U(\varepsilon)$  is independent of  $\varepsilon$ .

Suppose that the bundles  $V^\pm$  are equipped with Hermitian inner products and that the operators  $D^\pm$  are formally self-adjoint. This means that the associated connections  $\nabla^\pm$  are unitary and the endomorphisms  $E^\pm$  are symmetric. Suppose



that  $U$  is self-adjoint. Let  $\phi := (\phi^+, \phi^-)$  and  $\psi := (\psi^+, \psi^-)$  satisfy transmittal boundary conditions. Since  $\nu$  is the inward unit normal of  $\Sigma \subset M^+$  and the outward unit normal of  $\Sigma \subset M^-$ , we may integrate by parts to show that  $D$  is self-adjoint by computing:

$$\begin{aligned}
& (D\phi, \psi)_{L^2} - (\phi, D\psi)_{L^2} \\
&= \int_{M^+} \{(\phi_{;ii}^+, \psi^+) - (\phi^+, \psi_{;ii}^+)\} + \int_{M^-} \{(\phi_{;ii}^-, \psi^-) - (\phi^-, \psi_{;ii}^-)\} \\
&= \int_{M^+} \{(\phi_{;i}^+, \psi^+) - (\phi^+, \psi_{;i}^+)\}_{;i} + \int_{M^-} \{(\phi_{;i}^-, \psi^-) - (\phi^-, \psi_{;i}^-)\}_{;i} \quad (17) \\
&= - \int_{\Sigma} \{(\phi_{;\nu}^+ - \phi_{;\nu}^-, \psi) - (\phi, \psi_{;\nu}^+ - \psi_{;\nu}^-)\} \\
&= - \int_{\Sigma} \{(U\phi, \psi) - (\phi, U\psi)\} = 0.
\end{aligned}$$

## 4 Invariance Theory

We begin by giving the proof of Lemma 2.4. We assign degree 1 to the tensors  $\{L^\pm, U, \omega\}$  and assign degree 2 to the tensors  $\{R^\pm, \Omega^\pm, E^\pm\}$ . We increment the degree by 1 for every explicit covariant derivative which appears. Dimensional analysis shows that the integrands  $a_n^\Sigma$  can be built universally and polynomial from monomials which are homogeneous of weighted degree  $n - 1$  and which are non-chiral. The structure group is  $O(m - 1)$ . We use H. Weyl's theorem on the invariants of the orthogonal group to write down a spanning set; product formulas then yield the coefficients are dimension free except for the normalizing factor of  $(4\pi)^{-m/2}$ .  $\square$

Thus to determine the formulas for the  $a_n^\Sigma$ , we must determine the unknown coefficients in Lemma 2.4. We shall use the various functorial properties of these invariants in the calculation. We begin our evaluation with:

**Lemma 4.1** *We have  $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$ .*

**Proof:** Suppose we take  $U = 0$  and let  $(f, D)$  be smooth on all of  $M$ . Then the hypersurface  $\Sigma$  plays no role and thus the invariants  $a_n^\Sigma$  vanish in this setting. The terms indexed by these coefficients  $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ , and  $c_8$  survive and thus these coefficients must vanish.  $\square$

## 5 Manifolds with boundary

Let  $M_0$  be a smooth Riemannian manifold with smooth boundary  $\partial M_0$  and let  $D_0$  be an operator of Laplace type over  $M_0$ . Let

$$\mathcal{B}_D\phi := \phi|_{\partial M_0} \text{ and } \mathcal{B}_S\phi := (\nabla_\nu\phi + S\phi)|_{\partial M_0}.$$

The operator  $\mathcal{B}_D$  defines Dirichlet boundary conditions and the operator  $\mathcal{B}_S$  defines Robin boundary conditions. Let  $D_{\mathcal{B}}$  be the realization of  $D$  with the associated boundary condition. If  $f$  is a smooth function on  $M$ , then

$$\begin{aligned} \text{Tr}_{L^2}(f e^{-tD_{\mathcal{B}_{D/S}}}) &\sim \sum_{n \geq 0} t^{(n-m)/2} a_n(f, D, \mathcal{B}_{D/S}) \text{ where} \\ a_n(f, D, \mathcal{B}_{D/S}) &= a_n^M(f, D) + a_n^{\partial M_0}(f, D, \mathcal{B}_{D/S}) \end{aligned}$$

are given by local formulas. The interior invariants  $a_n^M(f, D)$  can be calculated using Theorem 2.2. Formulas if  $n \leq 5$  are known for the invariants  $a_n^M(f, D, \mathcal{B}_{D/S})$  for  $n \leq 5$ ; see for example [10, 11, 12, 26, 27, 29, 30, 37]. These results yield the following:

**Lemma 5.1**

1.  $a_0^{\partial M}(f, D, \mathcal{B}_{D/S}) = 0.$
2.  $a_1^{\partial M}(f, D, \mathcal{B}_D) = -(4\pi)^{(1-m)/2} \frac{1}{4} \int_{\partial M} \text{Tr}(I).$
3.  $a_1^{\partial M}(f, D, \mathcal{B}_S) = (4\pi)^{(1-m)/2} \frac{1}{4} \int_{\partial M} \text{Tr}(I).$
4.  $a_2^{\partial M}(f, D, \mathcal{B}_D) = (4\pi)^{-m/2} \frac{1}{6} \int_{\partial M} \text{Tr} \{2f L_{aa} I - 3f_{;m} I\}.$
5.  $a_2^{\partial M}(f, D, \mathcal{B}_S) = (4\pi)^{-m/2} \frac{1}{6} \int_{\partial M} \text{Tr} \{f(2L_{aa} I + 12S) + 3f_{;m} I\}.$
6.  $a_3^{\partial M}(f, D, \mathcal{B}_D) = -(4\pi)^{(1-m)/2} \frac{1}{384} \int_{\partial M} \text{Tr} \{96fE + f(16R_{ijji} - 8R_{amma} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab})I - 30f_{;m}L_{aa}I + 24f_{;mm}I\}.$
7.  $a_3^{\partial M}(f, D, \mathcal{B}_S) = +(4\pi)^{(1-m)/2} \frac{1}{384} \int_{\partial M} \text{Tr} (96fE + f(16R_{ijji} - 8R_{amma} + 13L_{aa}L_{bb} + 2L_{ab}L_{ab})I + f(96SL_{aa} + 192S^2) + f_{;m}(6L_{aa}I + 96S) + 24f_{;mm}I\}.$
8.  $a_4^{\partial M}(f, D, \mathcal{B}_D) = (4\pi)^{-m/2} \frac{1}{360} \int_{\partial M} \text{Tr} \{f(-120E_{;m} + 120EL_{aa}) + f(-18R_{ijji;m} + 20R_{ijji}L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac} + 24L_{aa;bb} + \frac{40}{21}L_{aa}L_{bb}L_{cc} - \frac{88}{7}L_{ab}L_{ab}L_{cc} + \frac{320}{21}L_{ab}L_{bc}L_{ac})I - 180f_{;m}E + f_{;m}(-30R_{ijji} - \frac{180}{7}L_{aa}L_{bb} + \frac{60}{7}L_{ab}L_{ab})I + 24f_{;mm}L_{aa}I - 30f_{;iim}I\}.$
9.  $a_4^{\partial M}(f, D, \mathcal{B}_S) = (4\pi)^{-m/2} \frac{1}{360} \int_{\partial M} \text{Tr} \{f(240E_{;m} + 120EL_{aa}) + f(42R_{ijji;m} + 24L_{aa;bb} + 20R_{ijji}L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac} + \frac{40}{3}L_{aa}L_{bb}L_{cc} + 8L_{ab}L_{ab}L_{cc} + \frac{32}{3}L_{ab}L_{bc}L_{ac})I + f(720SE + 120SR_{ijji} + 144SL_{aa}L_{bb} + 48SL_{ab}L_{ab} + 480S^2L_{aa} + 480S^3 + 120S_{;aa}) + f_{;m}(180E + 72SL_{aa} + 240S^2) + f_{;m}(30R_{ijji} + 12L_{aa}L_{bb} + 12L_{ab}L_{ab})I + 120f_{;mm}S + 24f_{;mm}L_{aa}I + 30f_{;iim}I\}.$

We extend results of [9] for smooth metrics to the current setting to relate the invariants  $a_n^{\partial M}(f, D, \mathcal{B}_{D/S})$  to the invariants  $a_n^{\Sigma}(f, D)$  as follows.

**Lemma 5.2** *Let  $M^\pm$  be two copies of a smooth manifold  $M_0$  joined along the common boundary. Let  $D^\pm := D_0$  and let  $U = -2S$ . Extend  $f_0 \in C^\infty(M_0)$  to  $M$  as an even function  $f$ . Then*

$$a_n^\Sigma(f, D, U) = a_n^{\partial M_0}(f_0, D_0, \mathcal{B}_D) + a_n^{\partial M_0}(f_0, D_0, \mathcal{B}_S).$$

**Proof:** Let  $\{\lambda_{D,i}, \tilde{\phi}_{D,i}\}$  and  $\{\lambda_{S,i}, \tilde{\phi}_{S,i}\}$  be the discrete spectral resolutions of  $D_0$  for Dirichlet and Robin boundary conditions over  $M_0$ . We wish to use these collections to construct the discrete spectral resolution of  $D$  with the given transmittal boundary conditions. Extend the sections  $\tilde{\phi}_{D,i}$  to be odd sections and the  $\tilde{\phi}_{S,i}$  to be even sections on  $V$  over  $M$  by defining:

$$\phi_{D,i}(x^\pm) = \pm \frac{1}{\sqrt{2}} \tilde{\phi}_{D,i}(x) \text{ and } \phi_{S,i}(x^\pm) = \frac{1}{\sqrt{2}} \tilde{\phi}_{S,i}(x).$$

We show the sections  $\{\phi_{D,i}, \phi_{S,j}\}$  form an orthonormal system by computing:

$$\begin{aligned} \int_M (\phi_{D,i}, \phi_{D,j}) &= \frac{1}{2} \int_{M^+} (\tilde{\phi}_{D,i}, \tilde{\phi}_{D,j}) + (-1)^2 \frac{1}{2} \int_{M^-} (\tilde{\phi}_{D,i}, \tilde{\phi}_{D,j}) = \delta_{ij}, \\ \int_M (\phi_{D,i}, \phi_{S,j}) &= \frac{1}{2} \int_{M^+} (\tilde{\phi}_{D,i}, \tilde{\phi}_{S,j}) - \frac{1}{2} \int_{M^-} (\tilde{\phi}_{D,i}, \tilde{\phi}_{S,j}) = 0, \text{ and} \\ \int_M (\phi_{S,i}, \phi_{S,j}) &= \frac{1}{2} \int_{M^+} (\tilde{\phi}_{S,i}, \tilde{\phi}_{S,j}) + \frac{1}{2} \int_{M^-} (\tilde{\phi}_{S,i}, \tilde{\phi}_{S,j}) = \delta_{ij}. \end{aligned}$$

Let  $\phi = (\phi^+, \phi^-)$ . We define:

$$\tilde{\phi}_e(x) = \frac{1}{2}(\phi^+(x^+) + \phi^-(x^-)) \text{ and } \tilde{\phi}_o(x) = \frac{1}{2}(\phi^+(x^+) - \phi^-(x^-)).$$

We expand  $\tilde{\phi}_{e/o}$  using the sections  $\phi_{S/D,j}$ . We then extend  $\tilde{\phi}_{e/o}$  to an even/odd pair of sections  $\phi_{e/o}$ . Since  $\phi = \phi_e + \phi_o$ , this shows that the sections  $\{\tilde{\phi}_{D,i}, \tilde{\phi}_{S,i}\}$  form a complete orthonormal basis for  $L^2(V)$ . Since the  $\phi_{S,i}$  are even sections, they are continuous. Since the  $\phi_{D,i}$  are odd sections which vanish on the common boundary, they are continuous as well. We verify that the transmittal boundary conditions are satisfied by computing:

$$\begin{aligned} (\phi_{D,i;\nu^+}^+)|_\Sigma &= \frac{1}{\sqrt{2}}(\tilde{\phi}_{D,i;\nu})|_\Sigma = (-\phi_{D,i;\nu^-})|_\Sigma + 0 = (\phi_{D,i;\nu^+}^-)|_\Sigma - 2S\phi_{D,i}|_\Sigma \\ (\phi_{S,i;\nu^+}^+)|_\Sigma &= \frac{1}{\sqrt{2}}(\tilde{\phi}_{S,i;\nu})|_\Sigma = \frac{1}{\sqrt{2}}(\tilde{\phi}_{S,i;\nu})|_\Sigma - 2\frac{1}{\sqrt{2}}(\tilde{\phi}_{S,i;\nu} + S\tilde{\phi}_{S,i})|_\Sigma \\ &= -\frac{1}{\sqrt{2}}(\tilde{\psi}_{S,i;\nu})|_\Sigma - 2S\psi_{S,i}|_\Sigma = -(\psi_{S,i;\nu^-}^-)|_\Sigma - 2S\psi_{S,i}|_\Sigma \\ &= (\psi_{S,i;\nu^+}^-)|_\Sigma - 2S\psi_{S,i}|_\Sigma. \end{aligned}$$

This shows that  $\{(\lambda_{i,D}, \phi_{i,D}), (\lambda_{j,S}, \phi_{j,S})\}$  gives the desired discrete spectral resolution. Since  $f$  is an even function, we use equation (15) to see:

$$\begin{aligned} \text{Tr}_{L^2}(f e^{-tD}) &= \sum_i e^{-t\lambda_{i,D}} \int_M f(\phi_{i,D}, \phi_{i,D}) + \sum_j e^{-t\lambda_{j,S}} \int_M f(\phi_{j,S}, \phi_{j,S}) \\ &= \sum_i e^{-t\lambda_{i,D}} \int_{M_0} f_0(\tilde{\phi}_{i,D}, \tilde{\phi}_{i,D}) + \sum_j e^{-t\lambda_{j,S}} \int_{M_0} f_0(\tilde{\phi}_{j,S}, \tilde{\phi}_{j,S}) \\ &= \text{Tr}_{L^2}(f_0 e^{-tD_0, \mathcal{B}_D}) + \text{Tr}_{L^2}(f_0 e^{-tD_0, \mathcal{B}_S}). \end{aligned}$$

The desired result now follows by equating terms in the asymptotic expansions. We note that were we to extend  $f_0$  as an *odd* function, then the terms would cancel instead of combining and we would get 0.  $\square$

The following result is an immediate consequence of Lemmas 5.1 and 5.2.

**Lemma 5.3** *We have  $d_1 = 2$ ,  $d_2 = 0$ ,  $d_3 = -6$ ,  $d_4 = \frac{3}{2}$ ,  $d_5 = 3$ ,  $d_6 = 9$ ,  $d_7 = 48$ ,  $d_8 = -24$ , and  $d_9 = -24$ .*

## 6 A Transmittal problem for the de Rham cplx

To evaluate the coefficient  $e_1$ , we use the local index theorem. We begin by constructing transmittal boundary conditions for the de Rham complex. We begin by recalling some facts concerning exterior  $\text{ext}$ , interior  $\text{int}$ , and Clifford multiplication  $\text{cl}$ . Let  $U$  be a cotangent vector. We can choose a local orthonormal frame so  $U = ce^1$ . Let  $1 \leq i_1 < \dots < i_p \leq m$ . Then

$$\begin{aligned} \text{ext}(U)e^{i_1} \wedge \dots \wedge e^{i_p} &= ce^1 \wedge e^{i_1} \wedge \dots \wedge e^{i_p} \text{ if } 1 < i_1, \\ \text{ext}(U)e^{i_1} \wedge \dots \wedge e^{i_p} &= 0 \text{ if } 1 = i_1, \\ \text{int}(U)e^{i_1} \wedge \dots \wedge e^{i_p} &= 0 \text{ if } 1 < i_1, \text{ and} \\ \text{int}(U)e^{i_1} \wedge \dots \wedge e^{i_p} &= ce^{i_2} \wedge \dots \wedge e^{i_p} \text{ if } 1 = i_1. \end{aligned}$$

Thus exterior multiplication adds an index and interior multiplication cancels an index if possible. Let  $\text{cl}(U) := \text{ext}(U) - \text{int}(U)$  denote Clifford multiplication;

$$\text{cl}(U_1)\text{cl}(U_2) + \text{cl}(U_2)\text{cl}(U_1) = -2(U_1, U_2)I.$$

We can write the exterior derivative  $d$  and the interior derivative  $\delta$  in the form:

$$\begin{aligned} d\phi^\pm &= \text{ext}(e^i)\nabla_{e_i}^\pm \phi^\pm, & \delta\phi^\pm &= -\text{int}(e^i)\nabla_{e_i}^\pm \phi^\pm, \text{ and} \\ (d + \delta)\phi^\pm &= \text{cl}(e^i)\nabla_{e_i}^\pm \phi^\pm. \end{aligned} \tag{18}$$

Let  $M = M^+ \cup_\Sigma M^-$ ; we give  $M$  the smooth structure defined in Section 3. Let  $V := \Lambda$  be the exterior algebra. Let  $\phi = (\phi^+, \phi^-)$  where  $\phi^\pm$  are smooth differential forms over  $M^\pm$ . We let  $\phi^\pm|_\Sigma$  be sections to the full exterior bundle; we do not set  $dx^m$  to zero. Let

$$\mathcal{B}_0\phi := \phi^+|_\Sigma - \phi^-|_\Sigma.$$

Thus  $\mathcal{B}_0\phi = 0$  if and only if  $\phi$  is continuous on  $\Sigma$ . We let  $\Delta := (d + \delta)^2$  with

$$\text{Domain}(\Delta) := \mathcal{D} := \{\phi : \mathcal{B}_0\phi = 0 \text{ and } \mathcal{B}_0(d + \delta)\phi = 0\}.$$

The following Lemma will be crucial for our analysis.

**Lemma 6.1**

1. We have  $\mathcal{B}_0\phi = 0$  if and only if  $((d + \delta)\phi, \psi)_{L^2} = (\phi, (d + \delta)\psi)_{L^2}$  for every  $\psi$  satisfying  $\mathcal{B}_0\psi = 0$ .
2. The operator  $\Delta := (d + \delta)^2$  with the domain  $\mathcal{D}$  is self-adjoint.
3. Let  $U := \text{cl}(e^m)\text{cl}(e^a)\omega_a$ . Then  $\phi \in \mathcal{D}$  if and only if  $\mathcal{B}_U\phi = 0$ .
4. Let  $\mathcal{L}_{ab} := (L_{ab}^+ + L_{ab}^-)$ . Then  $\omega_a = \mathcal{L}_{ab}(\text{ext}(e^m)\text{int}(e^b) + \text{int}(e^m)\text{ext}(e^b))$  and  $U = \mathcal{L}_{ab}\{\text{ext}(e^m)\text{int}(e^m)\text{int}(e^a)\text{ext}(e^b) + \text{int}(e^m)\text{ext}(e^m)\text{ext}(e^a)\text{int}(e^b)\}$ .
5. Since  $U\Lambda^p \subset \Lambda^p$ ,  $U$  induces transmittal boundary conditions for  $\Delta_p$ . We have  $a_n(1, \Delta_e, U_e) - a_m(1, \Delta_o, U_o) = 0$  for  $n \neq m$  and  $a_m(1, \Delta_e, U_e) - a_m(1, \Delta_o, U_o) \in \mathbb{Z}$ .
6. We have  $\dim \ker(\Delta_{\mathcal{B}_U}^e) - \dim \ker(\Delta_{\mathcal{B}_U}^o) = \chi(M)$ .

**Proof:** We apply equation (18) to study  $((d + \delta)\phi, \psi)_{L^2} - (\phi, (d + \delta)\psi)_{L^2}$ . We can integrate by parts to exchange tangential derivatives; these cancel automatically. We assume  $\psi^+|_\Sigma = \psi^-|_\Sigma$ . After taking into account the different signs of the relevant normals, Green's formula yields, modulo a possible sign convention that plays no role,

$$((d + \delta)\phi, \psi)_{L^2} - (\phi, (d + \delta)\psi)_{L^2} = \int_\Sigma \{\phi^+|_\Sigma - \phi^-|_\Sigma\} \cdot \{\text{cl}(\nu)\psi|_\Sigma\}. \quad (19)$$

Since  $(\text{cl} \nu)^2 = -1$ , the terms in equation (19) vanish for all suitable  $\psi$  if and only if  $\phi^+|_\Sigma = \phi^-|_\Sigma$ . If  $\phi, \psi \in \mathcal{D}$ , then we can use assertion (1) to show that  $D$  with this realization is self-adjoint by observing that:

$$\begin{aligned} (\Delta\phi, \psi)_{L^2} &= ((d + \delta)\phi, (d + \delta)\psi)_{L^2} \text{ since } \mathcal{B}_0(d + \delta)\phi = 0 \text{ and } \mathcal{B}_0\psi = 0 \\ (\phi, \Delta\psi)_{L^2} &= ((d + \delta)\phi, (d + \delta)\psi)_{L^2} \text{ since } \mathcal{B}_0\phi = 0 \text{ and } \mathcal{B}_0(d + \delta)\psi = 0. \end{aligned}$$

Let  $\mathcal{B}_0\phi = 0$ . We compute:

$$\begin{aligned} & \{(d + \delta^+)\phi^+\}|_\Sigma - \{(d + \delta^-)\phi^-\}|_\Sigma \\ &= \text{cl}(e^i)\{(\nabla_i^+\phi^+)|_\Sigma - (\nabla_i^-\phi^-)|_\Sigma\} \\ &= \text{cl}(e^a)\{\nabla_a^+ - \nabla_a^-\}\phi|_\Sigma + \text{cl}(e^m)\{(\nabla_\nu^+\phi^+)|_\Sigma - (\nabla_\nu^-\phi^-)|_\Sigma\} \\ &= \text{cl}(e^m)\{(\nabla_\nu^+\phi^+)|_\Sigma - (\nabla_\nu^-\phi^-)|_\Sigma - \text{cl}(e^m)\text{cl}(e^a)\omega_a\phi|_\Sigma\}. \end{aligned} \quad (20)$$

Since  $\text{cl}(e^m)$  is an isomorphism, the terms in (20) vanish if and only if  $\phi$  satisfies the transmittal boundary condition defined by  $U$ . We compute:

$$\begin{aligned} \nabla_i(e^{i_1} \wedge \dots \wedge e^{i_p}) &= \sum_{1 \leq j \leq p} (-1)^{j-1} \Gamma_{ii_j\ell} e^{i_1} \wedge \dots \wedge e^\ell \wedge \dots \wedge e^{i_p} \\ &= \Gamma_{ik\ell} \text{ext}(e^\ell) \text{int}(e^k) \\ \omega_a &= \mathcal{L}_{ab}\{\text{ext}(e^m)\text{int}(e^b) - \text{ext}(e^b)\text{int}(e^m)\} \\ &= \mathcal{L}_{ab}\{\text{ext}(e^m)\text{int}(e^b) + \text{int}(e^m)\text{ext}(e^b)\} \end{aligned}$$

If  $i \neq j$ , then

$$\begin{aligned} \text{ext}(e^i)\text{ext}(e^i) + \text{ext}(e^j)\text{ext}(e^j) &= 0, \\ \text{int}(e^i)\text{ext}(e^j) + \text{ext}(e^j)\text{int}(e^i), \text{ and} \\ \text{int}(e^i)\text{int}(e^j) + \text{int}(e^j)\text{int}(e^i) &= 0. \end{aligned}$$

Furthermore  $\text{ext}(e^i)\text{ext}(e^i) = 0$  and  $\text{int}(e^i)\text{int}(e^i) = 0$ . Consequently:

$$\begin{aligned} U &= \mathcal{L}_{ab}(\text{ext}(e^m) - \text{int}(e^m))(\text{ext}(e^a) - \text{int}(e^a)) \\ &\quad \cdot (\text{ext}(e^m)\text{int}(e^b) + \text{int}(e^m)\text{ext}(e^b)) \\ &= \mathcal{L}_{ab}\{\text{ext}(e^m)\text{int}(e^m)\text{int}(e^a)\text{ext}(e^b) \\ &\quad + \text{int}(e^m)\text{ext}(e^m)\text{ext}(e^a)\text{int}(e^b)\}. \end{aligned}$$

We use assertion (4) to see that  $U\Lambda^p \subset \Lambda^p$  since there are two  $\text{ext}$  and two  $\text{int}$  terms. It is also clear that  $U$  is self-adjoint; this gives another proof of assertion (2). We extend the cancellation argument of McKean and Singer [28] to prove assertion (5). Let  $E(\lambda, \Delta, U)$  be the associated eigenspaces. Suppose  $\Delta\phi = \lambda\phi$  and that  $\mathcal{B}_U\phi = 0$ . Since  $(d + \delta)\Delta = \Delta(d + \delta)$ , we have  $\Delta(d + \delta)\phi = \lambda(d + \delta)\phi$ . We show that  $(d + \delta)\phi \in \mathcal{D}$  by computing:

$$\mathcal{B}_0(d + \delta)\phi = 0 \text{ and } \mathcal{B}_0(d + \delta)(d + \delta)\phi = \lambda\mathcal{B}_0\phi = 0.$$

Thus we have  $(d + \delta) : E(\lambda, \Delta_{e/o}, U_{e/o}) \rightarrow E(\lambda, \Delta_{o/e}, U_{o/e})$ . If  $\lambda \neq 0$ , then  $(d + \delta)^2 = \lambda$  is an isomorphism and thus

$$\dim(E(\lambda, D_e, U_e)) - \dim(E(\lambda, D_o, U_o)) = 0 \text{ for } \lambda \neq 0. \quad (21)$$

We use equation (21) to compute:

$$\text{Tr}_{L^2}(e^{-tD_U^e}) - \text{Tr}_{L^2}(e^{-tD_U^o}) \quad (22)$$

$$\begin{aligned} &= \sum_{\lambda} e^{-t\lambda} \{\dim(E(\lambda, \Delta_e, U_e)) - \dim(E(\lambda, \Delta_o, U_o))\} \\ &= \dim(E(0, \Delta_e, U_e)) - \dim(E(0, \Delta_o, U_o)). \end{aligned} \quad (23)$$

We compare coefficients of powers of  $t$  in the asymptotic expansion on the left in (22) with (23) to see that the constant term is an integer and the other terms vanish.

It now follows that the index is given by a local formula and thus is constant under deformations. We fix the metric on  $\Sigma$  and deform the metrics on  $M^{\pm}$  so that the metric is product near the boundary. We then have  $U = 0$  and  $\Sigma$  no longer plays a role. Thus the index is given by  $\int_M \{a_m^M(\Delta^e) - a_m^M(\Delta^o)\}$  and the standard local index theorem shows the index to be the Euler-Poincare characteristic  $\chi(M)$ .  $\square$

We can now determine the remaining unknown coefficient:

**Lemma 6.2** *We have  $e_1 = 24$ .*

**Proof:** We apply Lemma 6.1 with  $m = 2$ . Invariants which are multiplied by  $\text{Tr}(I)$  cancel in the alternating sum. We set  $f = 1$  so the derivatives of  $f$  play no role. Thus the only terms which survive involve  $U$  and  $\omega_a$  on  $\Lambda^1$ . Let  $\mathcal{L} := \mathcal{L}_{11}$ . We use Lemma 6.1 to see that:

$$\begin{aligned}\omega_1(1) &= 0, \quad \omega_1(e^1 \wedge e^2) = 0, \quad \omega_1(e^1) = \mathcal{L}e^2, \quad \omega_1(e^2) = -\mathcal{L}e^1, \\ U_0 &= 0, \quad U_2 = 0, \quad U_1(e^1) = \mathcal{L}e^1, \quad U_1(e^2) = \mathcal{L}e^2, \\ (L_{aa}^+ + L_{aa}^-)\text{Tr}(U) &= 2\mathcal{L}^2, \quad \text{Tr}(U^2) = 2\mathcal{L}^2, \quad \text{and} \quad \text{Tr}(\omega_a \omega_a) = -2\mathcal{L}^2\end{aligned}$$

We use Lemma 6.1 (5) to see that:

$$\int_{\Sigma} \text{Tr}(e_1 \omega_1 \omega_1 + 48U_1^2 - 24(L_{11}^+ + L_{11}^-)U_1) = 0.$$

Consequently  $-2e_1 + 96 - 48 = 0$  so  $e_1 = 24$ .  $\square$

## 7 Computation of the fourth order invariant

In this section, we study the fourth order invariant. Before writing down a spanning set for the space of invariants, we make the following observations.

Certain invariants have been omitted because they are chiral - i.e. they change sign if we interchange the roles of  $+$  and  $-$ . We omit these invariants - typical examples would be

$$\text{Tr}\{fU\omega_{a;a}, f(L_{aa}^+ + L_{aa}^-)\omega_{b;b}\}.$$

If  $V^\pm$  are real vector bundles and if the operators  $D^\pm$  are real operators, then the invariants are real. Consequently, the coefficients are real. We suppose given fiber metrics on the bundles  $V^\pm$  and we suppose that the operators  $D^\pm$  are formally self-adjoint. Let  $U$  be self-adjoint. The calculations of equation (17) then show  $D$  is self-adjoint so again the invariants are real. Since  $\text{Tr}(\Omega)$  and  $\text{Tr}(\omega_a)$  are purely imaginary if  $D$  is self-adjoint, this observation shows that the following invariants play no role in the computation of  $a_4$ :

$$\begin{aligned}\{f\text{Tr}(\Omega_{a\nu^+;a}^+ + \Omega_{a\nu^-;a}^-), f(L_{bb}^+ - L_{bb}^-)\text{Tr}(\omega_{a;a}), f(L_{ab}^+ - L_{ab}^-)\text{Tr}(\omega_{a;b}), \\ f(L_{ab:b}^+ - L_{ab:b}^-)\text{Tr}(\omega_a), f(L_{bb:a}^+ - L_{bb:a}^-)\text{Tr}(\omega_a), f(\rho_{am}^+ - \rho_{am}^-)\text{Tr}(\omega_a), \\ f(L_{bb:a}^+ - L_{bb:a}^-)\text{Tr}(\omega_a), f(L_{ab:b}^+ - L_{ab:b}^-)\text{Tr}(\omega_a)\}.\end{aligned}$$

We can formulate now the main result of this section.

**Theorem 7.1** 1. *There exist universal constants  $\mathbf{b} = (b_1, \dots, b_{20})$  such that*

$$a_4^\Sigma(f, D, U) = (4\pi)^{-m/2} 360^{-1} \int_\Sigma \text{Tr}(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3)$$

where

$$\begin{aligned} \mathcal{A}_1 = & b_1(E^+ - E^-)(f_{;\nu^+} - f_{;\nu^-}) + b_2(\tau^+ - \tau^-)(f_{;\nu^+} - f_{;\nu^-}) \\ & + b_3(R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-})(f_{;\nu^+} - f_{;\nu^-}) + b_4f(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-) \\ & \times (L_{cc}^+ + L_{cc}^-) + b_5f(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(L_{cc}^+ + L_{cc}^-) + b_6f(L_{ab}^+ + L_{ab}^-) \\ & \times (L_{ab}^+ - L_{ab}^-)(L_{cc}^+ - L_{cc}^-) + b_7f(L_{ab}^+ - L_{ab}^-)(L_{bc}^+ - L_{bc}^-)(L_{ca}^+ + L_{ca}^-) \\ & + b_8(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) + b_9(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-) \\ & \times (f_{;\nu^+} + f_{;\nu^-}) + b_{10}(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\ & + b_{11}(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) + b_{12}f(E^+ - E^-)(L_{aa}^+ - L_{aa}^-) \\ & + b_{13}f(\tau^+ - \tau^-)(L_{aa}^+ - L_{aa}^-) + b_{14}f(R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-}) \\ & \times (L_{bb}^+ - L_{bb}^-) + b_{15}f(R_{a\nu^+b\nu^+} - R_{a\nu^-b\nu^-})(L_{ab}^+ - L_{ab}^-) \\ & + b_{16}f(R_{abcb}^+ - R_{abcb}^-)(L_{ac}^+ - L_{ac}^-) + b_{17}(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+\nu^+} - f_{;\nu^-\nu^-}) \\ & + b_{18}\omega_a^2(f_{;\nu^+} + f_{;\nu^-}) + b_{19}f\omega_a^2(L_{bb}^+ + L_{bb}^-) + b_{20}f\omega_a\omega_b(L_{ab}^+ + L_{ab}^-) \\ \mathcal{A}_2 = & 60f(E_{;\nu^+} + E_{;\nu^-}) + 12f(\tau_{;\nu^+} + \tau_{;\nu^-}) + 0(E^+ + E^-)(f_{;\nu^+} + f_{;\nu^-}) \\ & + 0(\tau^+ + \tau^-)(f_{;\nu^+} + f_{;\nu^-}) + 0(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-})(f_{;\nu^+} + f_{;\nu^-}) \\ & + 0[(\Delta f)_{;\nu^+} + (\Delta f)_{;\nu^-}] - 60f\omega_a(\Omega_{a\nu^+} - \Omega_{a\nu^-}) \\ & + \frac{40}{21}f(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(L_{cc}^+ + L_{cc}^-) - \frac{4}{7}f(L_{ab}^+ + L_{ab}^-) \\ & \times (L_{ab}^+ + L_{ab}^-)(L_{cc}^+ + L_{cc}^-) + \frac{68}{21}f(L_{ab}^+ + L_{ab}^-)(L_{bc}^+ + L_{bc}^-)(L_{ca}^+ + L_{ca}^-) \\ & - \frac{12}{7}(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) + \frac{18}{7}(L_{ab}^+ + L_{ab}^-) \\ & \times (L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) + 24f(L_{aa:bb}^+ + L_{aa:bb}^-) \\ & + 0f(L_{ab:ab}^+ + L_{ab:ab}^-) + 60f(E^+ + E^-)(L_{aa}^+ + L_{aa}^-) \\ & + 10f(\tau^+ + \tau^-)(L_{aa}^+ + L_{aa}^-) + 2f(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-})(L_{aa}^+ + L_{aa}^-) \\ & - 6f(R_{a\nu^+b\nu^+} + R_{a\nu^-b\nu^-})(L_{ab}^+ + L_{ab}^-) + 2f(R_{abcb}^+ + R_{abcb}^-)(L_{ac}^+ + L_{ac}^-) \\ & + 12(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+\nu^+} + f_{;\nu^-\nu^-}) \\ \mathcal{A}_3 = & -60fU^3 - 30fU(\tau^+ + \tau^-) - 180fU(E^+ + E^-) - 60fU_{;aa} \\ & + 0fU(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-}) + 15U(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+} - f_{;\nu^-}) \\ & - 9U(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+} + f_{;\nu^-}) + 0fU(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-) \\ & + 0fU(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-) - 18fU(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-) \\ & - 6fU(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ + L_{ab}^-) - 30U(f_{;\nu^+\nu^+} + f_{;\nu^-\nu^-}) \\ & + 30U^2(f_{;\nu^+} + f_{;\nu^-}) + 60fU^2(L_{aa}^+ + L_{aa}^-) - 60fU\omega_a^2 \end{aligned}$$

2. *The universal constants are given by*



$b_1 = -30$	$b_2 = -5$	$b_3 = 2$	$b_4 = 0$	$b_5 = -1$
$b_6 = -1$	$b_7 = 2$	$b_8 = 0$	$b_9 = 0$	$b_{10} = -5$
$b_{11} = -1$	$b_{12} = 0$	$b_{13} = 0$	$b_{14} = 0$	$b_{15} = 0$
$b_{16} = 2$	$b_{17} = 0$	$b_{18} = 18$	$b_{19} = 12$	$b_{20} = 24$

**Proof:** We can use the analysis of section 4 and the list of invariants which cannot contribute to  $a_4^\Sigma$  which was given at the beginning of this section to determine the general form of the invariant  $a_4^\Sigma$ . Coefficients of the invariants contained in  $\mathcal{A}_3$  have been calculated in [9, 29]. The coefficients listed in  $\mathcal{A}_2$  follow from Lemma 5.2 and the heat trace asymptotics for Dirichlet and Neumann boundary conditions (see Lemma 5.1).

Next we use the conformal properties of the heat kernel coefficients<sup>3</sup> [10].

**Lemma 7.2** *Let  $D(\epsilon) = e^{-2\epsilon f} D$ . We then have that*

$$\frac{d}{d\epsilon}|_{\epsilon=0} a_n(1, D(\epsilon)) = (m - n) a_n(f, D(0)).$$

We suppose that the conformal transformation parameter  $f$  is continuous but not necessarily smooth across  $\Sigma$ . The metric transforms as  $g(\epsilon) = e^{2f\epsilon} g$ . We have the following relations; a more extensive list is given in [10, 12], conformal variations of all invariants relevant for calculation of  $a_4^\Sigma$  are listed in Appendix:

$$\begin{aligned} \frac{d}{d\epsilon}|_{\epsilon=0} L_{ab}^\pm &= -f L_{ab}^\pm - f_{;\nu^\pm} \delta_{ab}, \\ \frac{d}{d\epsilon}|_{\epsilon=0} E^\pm &= -2f E^\pm + \frac{1}{2}(m - 2) f_{;\nu^\pm}, \\ \frac{d}{d\epsilon}|_{\epsilon=0} U &= -f U - \frac{1}{2}(m - 2)(f_{;\nu^+} + f_{;\nu^-}). \end{aligned}$$

We put  $n = 4$  in Lemma 7.2 and collect the terms with  $(E^+ - E^-)(f_{;\nu^+} - f_{;\nu^-})$ ,  $\omega_a^2(f_{;\nu^+} + f_{;\nu^-})$ ,  $(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-})$ ,  $(\tau^+ - \tau^-)(f_{;\nu^+} - f_{;\nu^-})$ ,  $(R_{a\nu^+ a\nu^+} - R_{a\nu^- a\nu^-})(f_{;\nu^+} - f_{;\nu^-})$ ,  $(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+ \nu^+} - f_{;\nu^- \nu^-})$ ,  $(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-})$ ,  $(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-})$ , and  $(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-})$  to obtain, respectively,

$$0 = -2(m - 1)b_{12} - 60(m - 4) - 2(m - 4)b_1, \quad (24)$$

$$0 = -30(2 - m) - (m - 1)b_{19} - b_{20} - (m - 4)b_{18}, \quad (25)$$

$$\begin{aligned} 0 &= 2(m - 1)b_4 + 2b_5 + b_6 + \frac{1}{4}(m - 2)b_{12} \\ &\quad - (m - 1)b_{13} + \frac{1}{2}b_{14} + \frac{1}{2}b_{16} + 15(m - 2) \\ &\quad - 10(m - 1) + 2 + (m - 4)b_{10}, \end{aligned} \quad (26)$$

$$0 = -5m + 18 - (m - 1)b_{13} + b_{16} - (m - 4)b_2, \quad (27)$$

---

<sup>3</sup>Note that we define the conformal transformations in such a way to make  $D$  conformally covariant. These transformations do not necessarily coincide with the conformal (Weyl) transformations adopted in physics.

$$0 = 2(m-6) - (m-1)b_{14} - b_{15} + 2b_{16} - (m-4)b_3, \quad (28)$$

$$0 = \frac{1}{2}(m-2)b_{12} - 2(m-1)b_{13} + (m-1)b_{14} + b_{15} - (m-4)b_{17}, \quad (29)$$

$$0 = -(m-1)b_4 - b_6 - \frac{1}{4}(m-2)b_{12} + (m-1)b_{13} - \frac{1}{2}b_{14} - \frac{1}{2}b_{16} - (m-4)b_8, \quad (30)$$

$$0 = -(m-1)b_5 - b_7 - \frac{1}{2}b_{15} - \frac{1}{2}(m-3)b_{16} - (m-4)b_9, \quad (31)$$

$$0 = -(m-1)b_6 - 2b_7 + 3 - \frac{1}{2}b_{15} - (m-3) - \frac{1}{2}(m-3)b_{16} - 2(m-4)b_{11}. \quad (32)$$

Since the universal constants  $b_i$  do not depend upon dimensions  $m$  we obtain from eq. (24):

$$b_{12} = 0, \quad b_1 = -30 \quad (33)$$

Next we consider the de Rham complex and use Lemma 6.1 to calculate

**Lemma 7.3**  $b_{19} + b_{20} = 36$ .

We omit the proof as it goes along exactly the same lines as that used to prove Lemma 6.2.

Together with the equation (25) above Lemma 7.3 gives:

$$b_{18} = 18, \quad b_{19} = 12, \quad b_{20} = 24. \quad (34)$$

The last ingredient which we use in this section is the following special case calculation:

**Lemma 7.4** *Let  $M^+$  be a unit hemisphere and  $M^-$  be a unit ball. Let  $D^\pm$  be a scalar Laplacian with  $E^+ = -\frac{1}{4}(m-1)^2$  and  $E^- = 0$ . Let  $U = (m-2)/2$ . Then*

$$360\Gamma(m/2)2^{m-1}a_4^\Sigma(1, D, U) = -\frac{250}{7} + \frac{2839}{42}m - \frac{191}{7}m^2 + \frac{61}{21}m^3.$$

**Proof:** By applying transmittal boundary conditions to eigenfunctions of the Laplace operator on the hemisphere and on the ball we obtain the following implicit equation for the eigenvalues  $\lambda$ :

$$\lambda J'_{l+(m-2)/2}(\lambda) P_{\lambda-1/2}^{-(l+(m-2)/2)}(0) - J_{l+(m-2)/2}(\lambda) \frac{d}{dx} P_{\lambda-1/2}^{-(l+(m-2)/2)}(x)|_{x=0} = 0, \quad (35)$$

where  $J$  and  $P$  are the Bessel and associated Legendre functions respectively. The heat kernel asymptotics is now calculated by applying the technique of [7, 8]. We do not give here details of this lengthy calculation.  $\square$

Lemma 7.4 and equations (26) - (32) fix the remaining universal constants. This completes the proof of Theorem 7.1.  $\square$

Let us mention, that by exploiting all conformal relations the numerical multipliers occuring in  $\mathcal{A}_3$  have been fully confirmed.

## 8 $a_5$ and renormalization of the brane-world scenario

A complete calculation of the fifth order term is hardly possible. Therefore, we restrict ourselves to a particular case relevant for a discussion of the divergences in the brane-world model. We suppose that the background field configuration is approximately symmetric under reflection about the surface  $\Sigma$ . This class of problems includes the standard brane described by the metric (4) together with some reasonable generalizations.

**Lemma 8.1** *Let all left and right limits of all non-chiral invariants up to dimension four coincide on the surface  $\Sigma$  while such limits of all chiral invariants up to dimension four change sign. Then*

$$\begin{aligned}
a_5(1, D, U) = & \frac{1}{5760} (4\pi)^{-(m-1)/2} \int_{\Sigma} \text{Tr} \{ -720 E_{;\nu} U + 120 \tau U^2 - 135 \tau_{;\nu} U \\
& + 30 \rho_{\nu\nu} U^2 + 240 U U_{;aa} + 720 E U^2 + 90 U^4 + 450 \Omega_{a\nu} \Omega_{a\nu} \\
& + 540 L_{aa} E_{;\nu} + \frac{195}{2} L_{aa} \tau_{;\nu} + 30 L_{ab} R_{a\nu\nu b;\nu} - 135 L_{aa} U_{;bb} \\
& - \frac{195}{4} L_{aa;c} L_{bb;c} - \frac{75}{2} L_{ab;c} L_{ab;c} + 30 L_{ab;a} L_{bc;c} - 720 L_{aa} E U \\
& - 15 L_{aa} U \rho_{\nu\nu} - 120 L_{aa} U \tau + 30 L_{ab} \rho_{ab} U - 90 L_{ab} U R_{a\nu\nu b} \\
& + 90 L_{aa} L_{bb} E + 180 L_{ab} L_{ab} E + 15 L_{aa} L_{bb} \tau + 30 L_{ab} L_{ab} \tau \\
& - \frac{15}{4} L_{aa} L_{bb} \rho_{\nu\nu} - \frac{15}{2} L_{ab} L_{ab} \rho_{\nu\nu} - 15 L_{cc} L_{ab} \rho_{ab} \\
& + 15 L_{cc} L_{ab} R_{a\nu\nu b} - 45 L_{ac} L_{ab} \rho_{bc} + 135 L_{ab} L_{ac} R_{b\nu\nu c} \\
& - 45 L_{ac} L_{db} R_{dacb} - \frac{315}{4} L_{cc} L_{ab} L_{ab} U - 75 L_{ab} L_{bc} L_{ac} U \\
& + 270 L_{aa} L_{bb} U^2 + 90 L_{ab} L_{ab} U^2 - \frac{885}{8} L_{aa} L_{bb} L_{cc} U - 270 L_{aa} U^3 \\
& + \frac{1053}{64} L_{aa} L_{bb} L_{cc} L_{dd} + \frac{279}{16} L_{cc} L_{dd} L_{ab} L_{ab} - \frac{921}{16} L_{ab} L_{ab} L_{cd} L_{cd} \\
& - \frac{57}{2} L_{dd} L_{ab} L_{bc} L_{ac} + \frac{639}{4} L_{ab} L_{bc} L_{cd} L_{ad} \} \tag{36}
\end{aligned}$$

We recall the definition of chirality which was given before Theorem 2.3. Proof of this Lemma follows immediately from Lemma 5.2 and the expressions for the  $a_5$  for Dirichlet and Robin boundary value problem [11, 27, 12].

If the operator  $D$  transforms covariantly under the Weyl rescalings, the coefficient  $a_5$  is proportional to the Weyl anomaly and can be used to derive the corresponding anomalous action.

The equation (36) represents the one-loop counterterms. A theory is multiplicatively renormalizable only if all independent counterterms are contained in the classical action. Of course, we cannot expect that a theory containing the

Einstein gravity will be multiplicatively renormalizable. One can hope, nevertheless, that renormalizability will be maintained at least in the matter sector. This seems however not easy to achieve. Consider for example the classical surface action of the form

$$S_{cl} = \int_{\Sigma} d^4x \sqrt{h} W(\phi) \quad (37)$$

Then the coefficient in front of the  $\delta$ -term in (6) is given by the second derivative of  $W$  with respect to  $\phi$ ,  $U \propto W''(\phi)$ . In the equation (36) we see a term  $U^4$ . Such a term should be contained in the classical action. This yields  $(W''(\phi))^4 \propto W(\phi)$ . This last condition is satisfied by a rather exotic potential  $W(\phi) \propto \phi^{8/3}$ . Phenomenological consequences of such a potential are quite unclear.

Certain simplifications could be achieved if one goes on shell, i.e. if it is supposed that the background fields satisfy their equations of motion. This will reduce the number of independent invariants. For example, the extrinsic curvature  $L_{ab}$  will be expressed by the Israel conditions [25] through the surface stress-energy tensor (which is essentially  $g_{ab}U$  in the simplified example (37)). Not much however can be gained on this way. First, going on shell has nothing to do with strict renormalization procedure of quantum field theory. Second, the number of divergent terms will be still considerable. As usual, supersymmetry leads to partial cancellation of the ultra violet divergences. For example, all terms of purely geometrical origin (i.e. without  $E$ ,  $U$  and  $\Omega$ ) will go away just due to the balance of bosonic and fermionic degrees of freedom. For recent work on supersymmetric brane-world scenario see [5, 6].

On the other hand, one may adopt a more radical and perhaps more fruitful point of view borrowed from string models. After separation of a few essential couplings, vanishing of the divergent field-dependent coefficients in front of these couplings could be considered as a restriction on the possible form of the (low-energy) background. Such restrictions may play a role of equations of motion for some effective theory. Practical realization of this scenario is far from being clear.

Finally we stress that the heat trace asymptotics is local. If there are more than one brane in the space-time the coefficients  $a_n$  are just sums of contributions of individual branes.

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## Appendix: Conformal variations

Here we list the conformal variations which have been used to obtain the relations (24)-(32). We integrate by parts where necessary to bring the variations into standard form so that  $f$  is not differentiated tangentially. We will be dealing with the terms  $X$  which are homogeneous of dimension 3. If  $f$  is constant, then  $\frac{d}{d\epsilon}|_{\epsilon=0}X = -3fX$ . To avoid writing the conformal weight repeatedly, we define  $\mathcal{C}X := \frac{d}{d\epsilon}|_{\epsilon=0}X + 3fX$ .

$$\begin{aligned}
b_4 \quad & \mathcal{C}(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(L_{cc}^+ + L_{cc}^-) = \\
& -2(m-1)(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -(m-1)(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
b_5 \quad & \mathcal{C}(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(L_{cc}^+ + L_{cc}^-) = \\
& -2(L_{aa}^+ - L_{aa}^-)(L_{cc}^+ + L_{cc}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -(m-1)(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
b_6 \quad & \mathcal{C}(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(L_{cc}^+ - L_{cc}^-) = \\
& -(L_{aa}^+ - L_{aa}^-)(L_{cc}^+ - L_{cc}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -(L_{aa}^+ + L_{aa}^-)(L_{cc}^+ - L_{cc}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -(m-1)(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
b_7 \quad & \mathcal{C}(L_{ab}^+ - L_{ab}^-)(L_{bc}^+ - L_{bc}^-)(L_{ca}^+ + L_{ca}^-) = \\
& -2(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
b_{12} \quad & \mathcal{C}(E^+ - E^-)(L_{aa}^+ - L_{aa}^-) = -(m-1)(E^+ - E^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& + \frac{1}{2}(m-2)(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+\nu^+} - f_{;\nu^-\nu^-}) \\
& - \frac{1}{4}(m-2)(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& - \frac{1}{4}(m-2)(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
b_{13} \quad & \mathcal{C}(\tau^+ - \tau^-)(L_{aa}^+ - L_{aa}^-) = -(m-1)(\tau^+ - \tau^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -2(m-1)(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+\nu^+} - f_{;\nu^-\nu^-}) \\
& +(m-1)(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& +(m-1)(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
b_{14} \quad & \mathcal{C}(R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-})(L_{bb}^+ - L_{bb}^-) = (m-1)(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+\nu^+} - f_{;\nu^-\nu^-}) \\
& - \frac{1}{2}(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& - \frac{1}{2}(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -(m-1)(R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-})(f_{;\nu^+} - f_{;\nu^-}) \\
b_{15} \quad & \mathcal{C}(R_{a\nu^+b\nu^+} - R_{a\nu^-b\nu^-})(L_{ab}^+ - L_{ab}^-) = -(R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-})(f_{;\nu^+} - f_{;\nu^-})
\end{aligned}$$

$$\begin{aligned}
& +(L_{aa}^+ - L_{aa}^-)(f_{;\nu^+\nu^+} - f_{;\nu^-\nu^-}) \\
& -\frac{1}{2}(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
b_{16} \quad & \mathcal{C}(R_{abcb}^+ - R_{abcb}^-)(L_{ac}^+ - L_{ac}^-) = 2(R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-})(f_{;\nu^+} - f_{;\nu^-}) \\
& -\frac{1}{2}(m-3)(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(m-3)(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& -\frac{1}{2}(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(L_{aa}^+ - L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& +(\tau^+ - \tau^-)(f_{;\nu^+} - f_{;\nu^-}) \\
b_{19} \quad & \mathcal{C}\omega_a^2(L_{bb}^+ + L_{bb}^-) = -(m-1)\omega_a^2(f_{;\nu^+} + f_{;\nu^-}) \\
b_{20} \quad & \mathcal{C}\omega_a\omega_b(L_{ab}^+ + L_{ab}^-) = -\omega_a^2(f_{;\nu^+} + f_{;\nu^-}) \\
60 \quad & \mathcal{C}(E^+ + E^-)(L_{aa}^+ + L_{aa}^-) = -(m-1)(E^+ + E^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& +\frac{1}{2}(m-2)(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+\nu^+} + f_{;\nu^-\nu^-}) + (m-2)f(L_{aa:bb}^+ + L_{aa:bb}^-) \\
& -\frac{1}{4}(m-2)(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{4}(m-2)(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
10 \quad & \mathcal{C}(\tau^+ + \tau^-)(L_{aa}^+ + L_{aa}^-) = -(m-1)(\tau^+ + \tau^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -2(m-1)(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+\nu^+} + f_{;\nu^-\nu^-}) - 4(m-1)f(L_{aa:bb}^+ \\
& + L_{aa:bb}^-) + (m-1)(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& + (m-1)(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
2 \quad & \mathcal{C}(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-})(L_{aa}^+ + L_{aa}^-) = (m-1)(L_{aa}^+ + L_{aa}^-)(f_{;\nu^+\nu^+} + f_{;\nu^-\nu^-}) \\
& -\frac{1}{2}(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) + 2f(L_{aa:bb}^+ + L_{aa:bb}^-) \\
& -(m-1)(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-})(f_{;\nu^+} + f_{;\nu^-}) \\
-6 \quad & \mathcal{C}(R_{a\nu^+b\nu^+} + R_{a\nu^-b\nu^-})(L_{ab}^+ + L_{ab}^-) = -(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-})(f_{;\nu^+} + f_{;\nu^-}) \\
& + (L_{aa}^+ + L_{aa}^-)(f_{;\nu^+\nu^+} + f_{;\nu^-\nu^-}) + 2f(L_{ab:ab}^+ \\
& + L_{ab:ab}^-) - \frac{1}{2}(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
2 \quad & \mathcal{C}(R_{abcb}^+ + R_{abcb}^-)(L_{ac}^+ + L_{ac}^-) = 2f(m-3)(L_{ab:ab}^+ + L_{ab:ab}^-)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(m-3)(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ + L_{ab}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(m-3)(L_{ab}^+ + L_{ab}^-)(L_{ab}^+ - L_{ab}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& +2f(L_{aa:bb}^+ + L_{aa:bb}^-) - \frac{1}{2}(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ + L_{bb}^-)(f_{;\nu^+} + f_{;\nu^-}) \\
& -\frac{1}{2}(L_{aa}^+ + L_{aa}^-)(L_{bb}^+ - L_{bb}^-)(f_{;\nu^+} - f_{;\nu^-}) \\
& +(\tau^+ + \tau^-)(f_{;\nu^+} + f_{;\nu^-}) + 2(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-})(f_{;\nu^+} + f_{;\nu^-}) \\
-60 \quad \mathcal{CU}\omega_a^2 = & -\frac{1}{2}(m-2)\omega_a^2(f_{;\nu^+} + f_{;\nu^-})
\end{aligned}$$

One should add the surface terms arising due to conformal variation of the bulk terms  $a^\pm(1, D)$  (see Theorem 2.2). The total derivative terms  $E_{;jj}$  and  $\tau_{;jj}$  in  $a_4(1, D)$  cancel the surface terms with  $(E_{;\nu^+}^+ + E_{;\nu^-}^-)$  and  $(\tau_{;\nu^+}^+ + \tau_{;\nu^-}^-)$ . Surface contributions of the conformal variation of the interior terms in  $M$  are:

$$\begin{aligned}
& \frac{1}{360}(4\pi)^{-m/2} \int_\Sigma \text{Tr} \{ (12m-48)f(\tau_{;\nu^+} + \tau_{;\nu^-}) + (-5m+18)((f_{;\nu^+} + f_{;\nu^-}) \\
& \quad \times (\tau^+ + \tau^-) - (f_{;\nu^+} - f_{;\nu^-})(\tau^+ - \tau^-)) + 60(m-4)f(E_{;\nu^+} + E_{;\nu^-}) \\
& \quad - 30(m-4)((f_{;\nu^+} + f_{;\nu^-})(E^+ + E^-) + (f_{;\nu^+} - f_{;\nu^-})(E^+ - E^-)) \\
& \quad + (2m-12)((f_{;\nu^+} + f_{;\nu^-})(R_{a\nu^+a\nu^+} + R_{a\nu^-a\nu^-}) + (f_{;\nu^+} - f_{;\nu^-}) \\
& \quad \times (R_{a\nu^+a\nu^+} - R_{a\nu^-a\nu^-})) + (4n-24)f(L_{aa:bb}^+ - L_{ab:ab}^+ + L_{aa:bb}^- - L_{ab:ab}^-) \}
\end{aligned}$$

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